

A Class of Linear-Quadratic-Gaussian (LQG) Mean-Field Game (MFG) of Stochastic Delay Systems

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Abstract

This paper investigates the linear-quadratic-Gaussian (LQG) mean-field game (MFG) for a class of stochastic delay systems. We consider a large population system in which the dynamics of each player satisfies some forward stochastic differential delay equation (SDDE). The consistency condition or Nash certainty equivalence (NCE) principle is established through an auxiliary mean-field system of anticipated forward-backward stochastic differential equation with delay (AFBSDDE). The wellposedness of such consistency condition system can be further established by some continuation method instead the classical fixed-point analysis. Thus, the consistency condition maybe given on arbitrary time horizon. The decentralized strategies are derived which are shown to satisfy the ϵ -Nash equilibrium property. Two special cases of our MFG for delayed system are further investigated.

Key words: Anticipated forward-backward stochastic differential equation with delay (AFBSDDE), Continuation method, ϵ -Nash equilibrium, Mean-field game, Stochastic differential equation with delay (SDDE).

1 Introduction

Recently, within the context of noncooperative game theory, the dynamic optimization of stochastic large-population system has attracted consistent and intense research attentions through a variety of fields including management science, engineering, mathematical finance and economics, social science, etc. The most special feature of controlled large-population system lies in the existence of considerable insignificant agents whose dynamics and (or) cost functionals are coupled via the state-average across the whole population. To design low-complexity strategies, one efficient methodology is the mean-field game (MFG) theory which enable us to obtain the decentralized control based on the individual own state together with some off-line quantity. The interested readers may refer [10, 17] for the motivation and methodology, and [1, 4–6] for recent progress in mean-field game theory. Besides, some other recent literature include [2, 3, 13–15, 18] for linear-quadratic-Gaussian (LQG) mean-field games of large-population system.

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It is remarkable that all agents in above literature are comparably negligible in that they are not able to affect the whole population in separable manner. By contrast, their impacts are imposed in an unified manner through the population state-average. In this sense, all agents can be viewed as negligible peers but they can generate some mass effects via some “unified manner” such as the control (input)-average or state (output)-average. These averages represent some type of impact imposed to other peers.

We point out in above works, all agents’ states are formulated by (forward) stochastic differential equations (SDEs) with the initial conditions as a priori. As a sequel, the agents’ objectives are minimizations of cost functionals involving their terminal states. In some realistic situation, there exist some phenomena in which the state behavior depends not only on the situation at time t , but also on a finite lagged state at $t - \theta$. Moreover, if we use the information which we know to anticipate the future evolution, we can get better results. As the novelty, this paper turns to consider the delay framework in which the agents’ dynamics is characterized by some (forward) stochastic differential equations with delay (SDDEs). It means that the impacts are hardly imposed to each agent immediately. A new type of BSDEs called anticipated BSDEs (ABSDEs) was introduced in [20], which type of BSDEs can be applied to many fields such as optimal control and finance. Based on it, the problems which depend not only the present but also the history were solved by [7]. In the consequent works, the FBSDEs with delay and related LQ problems were studied in [8] and [9]. A kind of stochastic maximum principle for optimal control problems of delay systems involving continuous and impulse controls was considered in [24]. The forward-backward linear quadratic stochastic optimal control problem with delay was investigated in [11]. And the maximum principle for optimal control of fully coupled forward stochastic differential delayed equations was derived in [12]. Moreover, some other important phenomena with delay were under consideration in [25, 26].

To formulate the above problem mathematically, some SDDE should be introduced to characterize the dynamics of the agents. It is remarkable that there exist rich literature concerning the theories and applications of SDDE. Generally, the large population problem with delay is under consideration. We discuss the related mean-field LQG games and derive the decentralized strategies. A stochastic process which relates to the delay term of control is introduced here to be the approximation of the control-average process. An auxiliary mean-field SDDE and a AFBSDDDE system are considered and analyzed. Here, the AFBSDDDE, which is composed by a SDDE and a ABSDE. Further, the AFBSDDDE can be divided into two simple AFBSDDDEs. In addition, the limit process is related to the wellposedness of a anticipated forward-backward ordinary differential equation with delay (AFBODDE) and a AFBSDDDE. We also derive the ϵ -Nash equilibrium property of decentralized control strategy with $\epsilon = O(1/\sqrt{N})$.

The rest of this paper is organized as follows. Section 2 formulates the large population LQG games of forward systems with delay. In Section 3, we derive the limiting optimal controls of the track systems and the consistency conditions. Section 4 is devoted to the related ϵ -Nash equilibrium property. Section 5 gives two special cases in this work.

2 Problem formulation

(Ω, \mathcal{F}, P) is a complete probability space on which a standard $(d + m \times N)$ -dimensional Brownian motion $\{W_t^0, W_t^i, 1 \leq i \leq N\}_{0 \leq t \leq T}$ is defined, in which a finite time horizon $[0, T]$ is considered for fixed $T > 0$. $\mathcal{F}_t^{W^0} := \sigma\{W_s^0, 0 \leq s \leq t\}$, $\mathcal{F}_t^{W^i} := \sigma\{W_s^i, 0 \leq s \leq t\}$, $\mathcal{F}_t^i := \sigma\{W_s^0, W_s^i; 0 \leq$

$s \leq t\}$. Here, $\{\mathcal{F}_t^{W^0}\}_{0 \leq t \leq T}$ stands for the common information of all players; while $\{\mathcal{F}_t^i\}_{0 \leq t \leq T}$ is the individual information of i^{th} player. Throughout this paper, \mathbb{R}^n denotes the n -dimensional Euclidean space, its usual norm $|\cdot|$ and the usual inner product $\langle \cdot, \cdot \rangle$. For a given vector or matrix M , M^\top stands for its transpose.

Moreover, we denote the spaces of matrices as follows.

- S^d : the space of all $d \times d$ symmetric matrices.
- S_+^d : the subspace of all positive semi-definite matrices of S^d .
- \hat{S}_+^d : the subspace of all positive definite matrices of S^d .

For any Euclidean space \mathbb{R}^n , we introduce the following notations:

- $L_{\mathcal{F}}^2(0, T; \mathbb{R}^n) = \{g : [0, T] \times \Omega \rightarrow \mathbb{R}^n \mid g(\cdot)$ is an \mathbb{R}^n -valued \mathcal{F}_t -progressively measurable process such that $\|g\|_{L_{\mathcal{F}}^2}^2 = \mathbb{E} \int_0^T |g(t)|^2 dt < \infty\}$.
- $L^2(0, T; \mathbb{R}^n) = \{g : [0, T] \rightarrow \mathbb{R}^n \mid g(\cdot)$ is an \mathbb{R}^n -valued deterministic function such that $\|g\|_{L^2}^2 = \int_0^T |g(t)|^2 dt < \infty\}$.
- $L^\infty(0, T; \mathbb{R}^n) = \{g : [0, T] \rightarrow \mathbb{R}^n \mid g(\cdot)$ is an \mathbb{R}^n -valued uniformly bounded function\}.
- $C(0, T; \mathbb{R}^n) = \{g : [0, T] \rightarrow \mathbb{R}^n \mid g(\cdot)$ is \mathbb{R}^n -valued continuous function\}.

In this paper, we consider a large population system with N individual agents, denoted by $\{\mathcal{A}_i\}_{1 \leq i \leq N}$. The dynamics of \mathcal{A}_i satisfies the following controlled stochastic differential equation with delay (SDDE):

$$\begin{cases} dx_t^i = \left[A_t x_t^i + \tilde{A}_t x_{t-\delta}^i + B_t u_t^i + \tilde{B}_t u_{t-\theta}^i + \frac{1}{N-1} \sum_{j=1, j \neq i}^N \hat{B}_t u_{t-\theta}^j \right] dt + \sigma_t dW_t^i + \sigma_t^0 dW_t^0, & t \in [0, T], \\ x_0^i = a, \quad x_t^i = \xi_t^i, \quad t \in [-\delta, 0), \quad u_t^i = \eta_t^i, \quad t \in [-\theta, 0), \end{cases} \quad (1)$$

where a is the initial state of \mathcal{A}_i , $x_{t-\delta}^i$ denotes the individual state delay, $u_{t-\theta}^i$ denotes the individual input or control delay. In addition, $\frac{1}{N-1} \sum_{j=1, j \neq i}^N \hat{B}_t u_{t-\theta}^j$ is introduced to denote the input delay of all other agents, imposed on a given agent \mathcal{A}_i . Similar state delay can be found in [22]. Here, for simplicity, we assume all agents are statistically identical (homogeneous) in that they share the same coefficients $(A, \tilde{A}, B, \tilde{B}, \hat{B}, \sigma, \sigma^0)$ and deterministic initial state a . The admissible control strategy $u^i \in \mathcal{U}_i$, where

$$\mathcal{U}_i := \left\{ u^i \mid u_t^i \in L_{\mathcal{F}_t^i}^2(0, T; \mathbb{R}^k) \right\}, \quad 1 \leq i \leq N.$$

Let $u = (u^1, \dots, u^N)$ denotes the set of strategies of all N agents; $u^{-i} = (u^1, \dots, u^{i-1}, u^{i+1}, \dots, u^N)$ the strategies set but excluding that of $\mathcal{A}_i, 1 \leq i \leq N$. Considering the state and control delay, the cost functional for $\mathcal{A}_i, 1 \leq i \leq N$ is given by

$$\begin{aligned} \mathcal{J}^i(u_t^i, u_t^{-i}) &= \frac{1}{2} \mathbb{E} \int_0^T \left[\langle R_t x_t^i, x_t^i \rangle + \langle \tilde{R}_t x_{t-\delta}^i, x_{t-\delta}^i \rangle + \langle N_t u_t^i, u_t^i \rangle + \langle \tilde{N}_t u_{t-\theta}^i, u_{t-\theta}^i \rangle \right] dt \\ &\quad + \frac{1}{2} \mathbb{E} \langle M x_T^i, x_T^i \rangle, \end{aligned} \quad (2)$$

where $\tilde{R}_t = 0$, $t \in [T, T + \delta]$, $\tilde{N}_t = 0$, $t \in [T, T + \theta]$.

For the coefficients of (1) and (2), we set the following assumption:

- (H1) $A_t, \tilde{A}_t \in L^\infty(0, T; \mathbb{R}^{n \times n})$, $B_t, \tilde{B}_t, \hat{B}_t \in L^\infty(0, T; \mathbb{R}^{n \times k})$, $\sigma_t \in L^2(0, T; \mathbb{R}^{n \times m})$, $\sigma_t^0 \in L^2(0, T; \mathbb{R}^{n \times d})$, $a \in \mathbb{R}^n$;
- (H2) $R_t, \tilde{R}_t \in L^\infty(0, T; S^n)$, $N_t, \tilde{N}_t \in L^\infty(0, T; S^k)$, and $R(\cdot) + \tilde{R}(\cdot + \delta) \in S_+^n$, for some $\delta > 0$; $N(\cdot) + \tilde{N}(\cdot + \theta) \in \hat{S}_+^n$ and the inverse $(N(\cdot) + \tilde{N}(\cdot + \theta))^{-1}$ is also bounded for some $\theta > 0$; $M \in S_+^n$.

Now, we formulate the large population dynamic optimization problem with delay.

Problem (LD). Find a control strategies set $\bar{u} = (\bar{u}^1, \dots, \bar{u}^N)$ which satisfies

$$\mathcal{J}^i(\bar{u}_t^i, \bar{u}_t^{-i}) = \inf_{u^i \in \mathcal{U}_i} \mathcal{J}^i(u_t^i, \bar{u}_t^{-i}), \quad 0 \leq i \leq N,$$

where \bar{u}^{-i} represents $(\bar{u}^1, \dots, \bar{u}^{i-1}, \bar{u}^{i+1}, \dots, \bar{u}^N)$, for $1 \leq i \leq N$.

3 The limiting optimal control and Nash certainty equivalence (NCE) equation system

To study Problem (LD), an efficient approach is to discuss the associated mean-field games by analyzing the asymptotic behavior when the agent number N tends to infinity. The key ingredient in this approach is to specify some suitable representation of state-average limit. With the help of such limit representation, we can figure out some auxiliary or tracking problem parameterized by the state-average limit. Based on it, the decentralized strategies of individual agents can thus be derived and we can also determine the state-average limit via some consistency condition. Moreover, the approximate Nash equilibrium property can be verified.

Noting that the agents are homogeneous, thus the optimal controls of \mathcal{A}_i , $1 \leq i \leq N$ are conditionally independent with identical distribution. Suppose $\frac{1}{N-1} \sum_{j=1, j \neq i}^N \hat{B}_t u_{t-\theta}^j$ is approximated by $m_0^\theta(t) \in \mathcal{F}_{t-\theta}^{W^0}$ as $N \rightarrow +\infty$. Introducing the following auxiliary dynamics of the players,

$$\begin{cases} dx_t^i = \left[A_t x_t^i + \tilde{A}_t x_{t-\delta}^i + B_t u_t^i + \tilde{B}_t u_{t-\theta}^i + m_0^\theta(t) \right] dt + \sigma_t dW_t^i + \sigma_t^0 dW_t^0, & t \in [0, T], \\ x_0^i = a, & x_t^i = \xi_t^i, \quad t \in [-\delta, 0), \quad u_t^i = \eta_t^i, \quad t \in [-\theta, 0). \end{cases} \quad (3)$$

The associated limiting cost functional becomes

$$\begin{aligned} J^i(u_t^i) &= \frac{1}{2} \mathbb{E} \int_0^T [\langle R_t x_t^i, x_t^i \rangle + \langle \tilde{R}_t x_{t-\delta}^i, x_{t-\delta}^i \rangle + \langle N_t u_t^i, u_t^i \rangle + \langle \tilde{N}_t u_{t-\theta}^i, u_{t-\theta}^i \rangle] dt \\ &\quad + \frac{1}{2} \mathbb{E} \langle M x_T^i, x_T^i \rangle. \end{aligned} \quad (4)$$

Thus, we formulate the limiting LQG game with delay (LLD) as follows.

Problem (LLD). To find an admissible control $\bar{u}^i \in \mathcal{U}_i$ for i^{th} agent \mathcal{A}_i satisfying

$$J^i(\bar{u}_t^i) = \inf_{u^i \in \mathcal{U}_i} J^i(u_t^i). \quad (5)$$

Such an admissible control \bar{u}^i is called an optimal control, and $\bar{x}^i(\cdot) = x_{\bar{u}}^i(\cdot)$ is called the corresponding optimal trajectory.

We link the Problem (LLD) to a stochastic Hamiltonian system as follows, which is an anticipated stochastic algebra differential equation system with delay,

$$\begin{cases} 0 = (N_t + \tilde{N}_{t+\theta})\bar{u}_t^i + B_t^\top \bar{y}_t^i + \tilde{B}_{t+\theta}^\top \mathbb{E}^{\mathcal{F}_t^i}[\bar{y}_{t+\theta}^i], \\ d\bar{x}_t^i = \left[A_t \bar{x}_t^i + \tilde{A}_t \bar{x}_{t-\delta}^i - B_t \bar{u}_t^i - \tilde{B}_t \bar{u}_{t-\theta}^i + m_0^\theta(t) \right] dt + \sigma_t dW_t^i + \sigma_t^0 dW_t^0, \\ d\bar{y}_t^i = - \left[A_t^\top \bar{y}_t^i + \tilde{A}_{t+\delta}^\top \mathbb{E}^{\mathcal{F}_t^i}[\bar{y}_{t+\delta}^i] + (R_t + \tilde{R}_{t+\delta}) \bar{x}_t^i \right] dt + \bar{z}_t^i dW_t^i + \bar{z}_t^0 dW_t^0, \quad t \in [0, T], \\ \bar{x}_0^i = a, \quad \bar{x}_t^i = \xi_t^i, \quad t \in [-\delta, 0), \quad u_t^i = \eta_t^i, \quad t \in [-\theta, 0), \\ \bar{y}_T^i = M \bar{x}_T^i, \quad \bar{y}_t^i = 0, \quad t \in (T, T + (\delta \vee \theta)]. \end{cases} \quad (6)$$

To get the optimal control of Problem (LLD), we have the following theorem.

Theorem 3.1 *Let (H1)-(H2) hold. The sufficient and necessary condition for the optimal control of \mathcal{A}_i for (LLD) is that u_t^i has the following form*

$$\bar{u}_t^i = -(N_t + \tilde{N}_{t+\theta})^{-1} (B_t^\top \bar{y}_t^i + \tilde{B}_{t+\theta}^\top \mathbb{E}^{\mathcal{F}_t^i}[\bar{y}_{t+\theta}^i]). \quad (7)$$

Moreover, for any given $m_0^\theta(t) \in L_{\mathcal{F}_{t-\theta}^{W^0}}^2(-\theta, T + (\delta \vee \theta); \mathbb{R}^n)$, the stochastic Hamiltonian system (6) admits a unique solution $(\bar{x}_t^i, \bar{u}_t^i, \bar{y}_t^i, \bar{z}_t^i, \bar{z}_t^0) \in L_{\mathcal{F}_t^i}^2(-\delta, T; \mathbb{R}^n) \times \mathcal{U}_i \times L_{\mathcal{F}_t^i}^2(-\theta, T + (\delta \vee \theta); \mathbb{R}^n) \times L_{\mathcal{F}_t^i}^2(0, T; \mathbb{R}^{n \times m}) \times L_{\mathcal{F}_t^i}^2(0, T; \mathbb{R}^{n \times d})$.

Proof The sufficient and necessary condition part could be from some variational calculus and dual representation, which is a straightforward consequence of the stochastic maximum principle in Yu [24]. We omit the proof.

Moreover, under assumption (H2), by the form of (7), our problem is to solve the following fully-coupled AFBSDE,

$$\begin{cases} d\bar{x}_t^i = \left[A_t \bar{x}_t^i + \tilde{A}_t \bar{x}_{t-\delta}^i - B_t (N_t + \tilde{N}_{t+\theta})^{-1} (B_t^\top \bar{y}_t^i + \tilde{B}_{t+\theta}^\top \mathbb{E}^{\mathcal{F}_t^i}[\bar{y}_{t+\theta}^i]) \right. \\ \quad \left. - \tilde{B}_t (N_{t-\theta} + \tilde{N}_t)^{-1} (B_{t-\theta}^\top \bar{y}_{t-\theta}^i + \tilde{B}_t \mathbb{E}^{\mathcal{F}_{t-\theta}^i}[\bar{y}_t^i]) + m_0^\theta(t) \right] dt + \sigma_t dW_t^i + \sigma_t^0 dW_t^0, \\ d\bar{y}_t^i = - \left[A_t^\top \bar{y}_t^i + \tilde{A}_{t+\delta}^\top \mathbb{E}^{\mathcal{F}_t^i}[\bar{y}_{t+\delta}^i] + (R_t + \tilde{R}_{t+\delta}) \bar{x}_t^i \right] dt + \bar{z}_t^i dW_t^i + \bar{z}_t^0 dW_t^0, \quad t \in [0, T], \\ \bar{x}_0^i = a, \quad \bar{x}_t^i = \xi_t^i, \quad t \in [-\delta, 0), \\ \bar{y}_T^i = M \bar{x}_T^i, \quad \bar{y}_t^i = 0, \quad t \in (T, T + (\delta \vee \theta)]. \end{cases} \quad (8)$$

Applying the classic “continuation method” which was proposed in [16], [21], the proof is similar as in the Appendix of [9], the above linear AFBSDE (8) has a unique solution. So the Hamiltonian system (6) admits a unique solution. \square

For the further studying, consider the following two AFBSDEs which are fully-coupled in states,

$$\begin{cases} dx_t^{i,1} = \left[A_t x_t^{i,1} + \tilde{A}_t x_{t-\delta}^{i,1} - B_t(N_t + \tilde{N}_{t+\theta})^{-1} (B_t^\top y_t^{i,1} + \tilde{B}_{t+\theta}^\top \mathbb{E}^{\mathcal{F}_t^{W^i}}[y_{t+\theta}^{i,1}]) \right. \\ \quad \left. - \tilde{B}_t(N_{t-\theta} + \tilde{N}_t)^{-1} (B_{t-\theta}^\top y_{t-\theta}^{i,1} + \tilde{B}_t^\top \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^i}}[y_t^{i,1}]) \right] dt + \sigma_t dW_t^i, \\ dy_t^{i,1} = - \left[A_t^\top y_t^{i,1} + \tilde{A}_{t+\delta}^\top \mathbb{E}^{\mathcal{F}_{t+\delta}^{W^i}}[y_{t+\delta}^{i,1}] + (R_t + \tilde{R}_{t+\delta}) x_t^{i,1} \right] dt + z_t^i dW_t^i, \quad t \in [0, T], \\ x_0^{i,1} = a^{i,1}, \quad x_t^{i,1} = \xi_t^{i,1}, \quad t \in [-\delta, 0), \\ y_T^{i,1} = M x_T^{i,1}, \quad y_t^{i,1} = 0, \quad t \in (T, T + (\delta \vee \theta)], \end{cases} \quad (9)$$

and

$$\begin{cases} dx_t^2 = \left[A_t x_t^2 + \tilde{A}_t x_{t-\delta}^2 - B_t(N_t + \tilde{N}_{t+\theta})^{-1} (B_t^\top y_t^2 + \tilde{B}_{t+\theta}^\top \mathbb{E}^{\mathcal{F}_t^{W^0}}[y_{t+\theta}^2]) \right. \\ \quad \left. - \tilde{B}_t(N_{t-\theta} + \tilde{N}_t)^{-1} (B_{t-\theta}^\top y_{t-\theta}^2 + \tilde{B}_t^\top \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^0}}[y_t^2]) + m_0^\theta(t) \right] dt + \sigma_t^0 dW_t^0, \\ dy_t^2 = - \left[A_t^\top y_t^2 + \tilde{A}_{t+\delta}^\top \mathbb{E}^{\mathcal{F}_{t+\delta}^{W^0}}[y_{t+\delta}^2] + (R_t + \tilde{R}_{t+\delta}) x_t^2 \right] dt + z_t^0 dW_t^0, \quad t \in [0, T], \\ x_0^2 = a^2, \quad x_t^2 = \xi_t^2, \quad t \in [-\delta, 0), \\ y_T^2 = M x_T^2, \quad y_t^2 = 0, \quad t \in (T, T + (\delta \vee \theta)], \end{cases} \quad (10)$$

where $a^i = a^{i,1} + a^2$, $\xi_t^i = \xi_t^{i,1} + \xi_t^2$. It follows from the Appendix in [9] that (9) and (10) admit the unique solutions $(x_t^{i,1}, y_t^{i,1}, z_t^i) \in L_{\mathcal{F}_t^{W^i}}^2(-\delta, T; \mathbb{R}^n) \times L_{\mathcal{F}_t^{W^i}}^2(-\theta, T + (\delta \vee \theta); \mathbb{R}^n) \times L_{\mathcal{F}_t^{W^i}}^2(0, T; \mathbb{R}^{n \times m})$ and $(x_t^2, y_t^2, z_t^0) \in L_{\mathcal{F}_t^{W^0}}^2(-\delta, T; \mathbb{R}^n) \times L_{\mathcal{F}_t^{W^0}}^2(-\theta, T + (\delta \vee \theta); \mathbb{R}^n) \times L_{\mathcal{F}_t^{W^0}}^2(0, T; \mathbb{R}^{n \times d})$. Then we have the following lemma.

Lemma 3.1 *Let (H1)-(H2) hold, if $(x_t^{i,1}, y_t^{i,1}, z_t^i)$ is the solution of (9) and (x_t^2, y_t^2, z_t^0) is the solution of (10), then $(x_t^{i,1} + x_t^2, y_t^{i,1} + y_t^2, z_t^i, z_t^0)$ is the solution of (8).*

Proof It is easily to check that $\bar{x}_t^i = x_t^{i,1} + x_t^2$, $\bar{y}_t^i = y_t^{i,1} + y_t^2$, $\bar{z}_t^i = z_t^i$ and $\bar{z}_t^0 = z_t^0$ are the solutions of AFBSDE (8), then we can get the conclusion. \square

In the following part, we will point out the essence of the limiting stochastic process $m_0^\theta(t)$. Firstly, we introduce the following AFBODDE and AFBSDE,

$$\begin{cases} d[\mathbb{E}x_t^1] = \left[A_t[\mathbb{E}x_t^1] + \tilde{A}_t[\mathbb{E}x_{t-\delta}^1] - B_t(N_t + \tilde{N}_{t+\theta})^{-1} (B_t^\top [\mathbb{E}y_t^1] + \tilde{B}_{t+\theta}^\top [\mathbb{E}y_{t+\theta}^1]) \right. \\ \quad \left. - \tilde{B}_t(N_{t-\theta} + \tilde{N}_t)^{-1} (B_{t-\theta}^\top [\mathbb{E}y_{t-\theta}^1] + \tilde{B}_t^\top [\mathbb{E}y_t^1]) \right] dt, \\ d[\mathbb{E}y_t^1] = - \left[A_t^\top [\mathbb{E}y_t^1] + \tilde{A}_{t+\delta}^\top [\mathbb{E}y_{t+\delta}^1] + (R_t + \tilde{R}_{t+\delta}) [\mathbb{E}x_t^1] \right] dt, \quad t \in [0, T], \\ \mathbb{E}x_0^1 = a^1, \quad \mathbb{E}x_t^1 = \mathbb{E}\xi_t^1, \quad t \in [-\delta, 0), \\ \mathbb{E}y_T^1 = M[\mathbb{E}x_T^1], \quad \mathbb{E}y_t^1 = 0, \quad t \in (T, T + (\delta \vee \theta)], \end{cases} \quad (11)$$

and

$$\left\{ \begin{aligned} dx_t^2 &= \left[A_t x_t^2 + \tilde{A}_t x_{t-\delta}^2 - B_t(N_t + \tilde{N}_{t+\theta})^{-1} (B_t^\top y_t^2 + \tilde{B}_{t+\theta}^\top \mathbb{E}^{\mathcal{F}_t^{W^0}}[y_{t+\theta}^2]) \right. \\ &\quad - (\tilde{B}_t + \hat{B}_t)(N_{t-\theta} + \tilde{N}_t)^{-1} (B_{t-\theta}^\top y_{t-\theta}^2 + \tilde{B}_t^\top \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^0}}[y_t^2]) \\ &\quad \left. - \hat{B}_t(N_{t-\theta} + \tilde{N}_t)^{-1} (B_{t-\theta}^\top [\mathbb{E} y_{t-\theta}^1] + \tilde{B}_t^\top [\mathbb{E} y_t^1]) \right] dt + \sigma_t^0 dW_t^0, \\ dy_t^2 &= - \left[A_t^\top y_t^2 + \tilde{A}_{t+\delta}^\top \mathbb{E}^{\mathcal{F}_t^{W^0}}[y_{t+\delta}^2] + (R_t + \tilde{R}_{t+\delta}) x_t^2 \right] dt + z_t^0 dW_t^0, \quad t \in [0, T], \\ x_0^2 &= a^2, \quad x_t^2 = \xi_t^2, \quad t \in [-\delta, 0), \\ y_T^2 &= M x_T^2, \quad y_t^2 = 0, \quad t \in (T, T + (\delta \vee \theta)]. \end{aligned} \right. \quad (12)$$

Proposition 3.1 $m_0^\theta(t)$ is in $L_{\mathcal{F}_t^{W^0}}^2(-\theta, T + (\delta \vee \theta); \mathbb{R}^n)$ and it is of the following form,

$$\begin{aligned} m_0^\theta(t) &= -\hat{B}_t(N_{t-\theta} + \tilde{N}_t)^{-1} [B_{t-\theta}^\top [\mathbb{E} y_{t-\theta}^1] + \tilde{B}_t^\top [\mathbb{E} y_t^1]] \\ &\quad - \hat{B}_t(N_{t-\theta} + \tilde{N}_t)^{-1} [B_{t-\theta}^\top y_{t-\theta}^2 + \tilde{B}_t^\top \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^0}}[y_t^2]], \end{aligned}$$

where y_t^1 is the solution of (11) and y_t^2 is the solution of (12)

Proof It follows from (9) and (10) that $y_t^{j,1}$ is independent of W_t^0 , y_t^2 is independent of W_t^j , for $1 \leq j \leq N$, respectively. Thus, we have

$$\mathbb{E}^{\mathcal{F}_{t-\theta}^j}[y_t^{j,1}] = \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^j}}[y_t^{j,1}], \quad \mathbb{E}^{\mathcal{F}_{t-\theta}^j}[y_t^2] = \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^0}}[y_t^2]. \quad (13)$$

By virtue of Lemma 3.1, we obtain

$$\begin{aligned} m_0^\theta(t) &= \lim_{N \rightarrow \infty} \hat{B}_t \frac{1}{N-1} \sum_{j=1, j \neq i}^N \bar{u}_{t-\theta}^j \\ &= -\hat{B}_t \lim_{N \rightarrow \infty} \frac{1}{N-1} \sum_{j=1, j \neq i}^N (N_{t-\theta} + \tilde{N}_t)^{-1} (B_{t-\theta}^\top \bar{y}_{t-\theta}^j + \tilde{B}_t^\top \mathbb{E}^{\mathcal{F}_{t-\theta}^j}[\bar{y}_t^j]) \\ &= -\hat{B}_t(N_{t-\theta} + \tilde{N}_t)^{-1} \lim_{N \rightarrow \infty} \frac{1}{N-1} \sum_{j=1, j \neq i}^N (B_{t-\theta}^\top (y_{t-\theta}^{j,1} + y_{t-\theta}^2) + \tilde{B}_t^\top \mathbb{E}^{\mathcal{F}_{t-\theta}^j}[(y_t^{j,1} + y_t^2)]) \\ &= -\hat{B}_t(N_{t-\theta} + \tilde{N}_t)^{-1} \left[\lim_{N \rightarrow \infty} \frac{1}{N-1} \sum_{j=1, j \neq i}^N (B_{t-\theta}^\top y_{t-\theta}^{j,1} + \tilde{B}_t^\top \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^j}}[y_t^{j,1}]) \right. \\ &\quad \left. + \lim_{N \rightarrow \infty} \frac{1}{N-1} \sum_{j=1, j \neq i}^N (B_{t-\theta}^\top y_{t-\theta}^2 + \tilde{B}_t^\top \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^0}}[y_t^2]) \right] \\ &= -\hat{B}_t(N_{t-\theta} + \tilde{N}_t)^{-1} [B_{t-\theta}^\top [\mathbb{E} y_{t-\theta}^{j,1}] + \tilde{B}_t^\top [\mathbb{E} y_t^{j,1}]] \\ &\quad - \hat{B}_t(N_{t-\theta} + \tilde{N}_t)^{-1} [B_{t-\theta}^\top y_{t-\theta}^2 + \tilde{B}_t^\top \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^0}}[y_t^2]] \\ &:= \Sigma_1^\theta(t) + \Sigma_2^\theta(t). \end{aligned} \quad (14)$$

Here, $\Sigma_1^\theta(t) = -\widehat{B}_t(N_{t-\theta} + \widetilde{N}_t)^{-1} [B_{t-\theta}^\top [\mathbb{E}y_{t-\theta}^{j,1}] + \widetilde{B}_t^\top [\mathbb{E}y_t^{j,1}]]$, which is the deterministic function; $\Sigma_2^\theta(t) = -\widehat{B}_t(N_{t-\theta} + \widetilde{N}_t)^{-1} [B_{t-\theta}^\top y_{t-\theta}^2 + \widetilde{B}_t^\top \mathbb{E}^{\mathcal{F}_{t-\theta}^{W_0}} [y_t^2]]$ is in $L_{\mathcal{F}_t^{W_0}}^2(-\theta, T + (\delta \vee \theta); \mathbb{R}^n)$. For $\mathbb{E}y_t^{i,1} = \mathbb{E}y_t^{j,1}$, for $i \neq j$, $i, j = 1, 2, \dots, N$, so $\mathbb{E}y_t^{i,1}$ is independent on i , then we denote $\mathbb{E}y_t^{i,1} = \mathbb{E}y_t^1$, where y_t^1 is the solution of (11). Thus $\Sigma_1^\theta(t)$ can be rewritten as

$$\Sigma_1^\theta(t) = -\widehat{B}_t(N_{t-\theta} + \widetilde{N}_t)^{-1} [B_{t-\theta}^\top [\mathbb{E}y_{t-\theta}^1] + \widetilde{B}_t^\top [\mathbb{E}y_t^1]].$$

Hence the result. \square

Remark 3.1 In what follows (11)-(12) are called the Nash certainty equivalence (NCE) equation system which can be used to determine the control state-average limit $m_0(t)$. Note that $m_0(t)$ plays an important role due to the dependence of decentralized strategy $\bar{u}_i(t)$ on it. We can see that \bar{u}_t^i in (7) is dependent on the solution \bar{y}_t^i and $\bar{y}_{t+\theta}^i$ of (6), and $\bar{y}_t^i, \bar{y}_{t+\theta}^i$ are dependent on $m_0(t)$.

4 ϵ -Nash equilibrium analysis

In above sections, we obtained the optimal control $\bar{u}_t^i, 1 \leq i \leq N$ of Problem (LLD) through the consistency condition system. Now, we turn to verify the ϵ -Nash equilibrium of Problem (LD). To start, we first present the definition of ϵ -Nash equilibrium.

Definition 4.1 A set of controls $u_t^i \in \mathcal{U}_i, 1 \leq i \leq N$, for N agents is called to satisfy an ϵ -Nash equilibrium with respect to the costs $J^i, 1 \leq i \leq N$, if there exists $\epsilon \geq 0$ such that for any fixed $1 \leq i \leq N$, we have

$$J^i(\bar{u}_t^i, \bar{u}_t^{-i}) \leq J^i(u_t^i, \bar{u}_t^{-i}) + \epsilon, \quad (15)$$

when any alternative control $u^i \in \mathcal{U}_i$ is applied by \mathcal{A}_i .

Remark 4.1 If $\epsilon = 0$, then Definition 4.1 is reduced to the usual Nash equilibrium.

Now, we state the main result of this paper and its proof will be given later.

Theorem 4.1 Under (H1)-(H2), $(\bar{u}_t^1, \bar{u}_t^2, \dots, \bar{u}_t^N)$ satisfies the ϵ -Nash equilibrium of (LD). Here, for $1 \leq i \leq N$, \bar{u}_t^i is given by (7).

The proof of Theorem 4.1 needs several lemmas which are presented later. Denoting \hat{x}_t^i is the centralized state trajectory with respect to \bar{u}_t^i ; \hat{x}_t^i is the decentralized one with respect to \bar{u}_t^i . The cost functionals for (LD) and (LLD) are denoted by $\mathcal{J}^i(\bar{u}_t^i, \bar{u}_t^{-i})$ and $J^i(\bar{u}_t^i)$, respectively.

Lemma 4.1

$$\sup_{1 \leq i \leq N} \left[\sup_{0 \leq t \leq T} \mathbb{E} \left| \frac{1}{N-1} \sum_{j=1, j \neq i}^N \widehat{B}_t \bar{u}_{t-\theta}^j - m_0^\theta(t) \right|^2 \right] = O\left(\frac{1}{N}\right), \quad (16)$$

where \bar{u}_t^j is given by (7).

Proof. By (7), (13) and Lemma 3.1, we get

$$\begin{aligned}
\frac{1}{N-1} \sum_{j=1, j \neq i}^N \widehat{B}_t \bar{u}_{t-\theta}^j &= \frac{1}{N-1} \sum_{j=1, j \neq i}^N \widehat{B}_t \left\{ - (N_{t-\theta} + \tilde{N}_t)^{-1} (B_{t-\theta}^\top \bar{y}_{t-\theta}^j + \tilde{B}_t^\top \mathbb{E}^{\mathcal{F}_{t-\theta}^j}[\bar{y}_t^j]) \right\} \\
&= -\widehat{B}_t (N_{t-\theta} + \tilde{N}_t)^{-1} \left\{ B_{t-\theta}^\top \left(\frac{1}{N-1} \sum_{j=1, j \neq i}^N y_{t-\theta}^{j,1} + y_{t-\theta}^2 \right) \right. \\
&\quad \left. + \tilde{B}_t^\top \left(\frac{1}{N-1} \sum_{j=1, j \neq i}^N \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^j}}[y_t^{j,1}] + \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^0}}[y_t^2] \right) \right\}. \tag{17}
\end{aligned}$$

Combining (17) and (14), we obtain

$$\begin{aligned}
&\frac{1}{N-1} \sum_{j=1, j \neq i}^N \widehat{B}_t \bar{u}_{t-\theta}^j - m_0^\theta(t) \\
&= -\widehat{B}_t (N_{t-\theta} + \tilde{N}_t)^{-1} \left\{ B_{t-\theta}^\top \left(\frac{1}{N-1} \sum_{j=1, j \neq i}^N y_{t-\theta}^{j,1} - \mathbb{E} y_{t-\theta}^1 \right) \right. \\
&\quad \left. + \tilde{B}_t^\top \left(\frac{1}{N-1} \sum_{j=1, j \neq i}^N \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^j}}[y_t^{j,1}] - \mathbb{E} y_t^1 \right) \right\}. \tag{18}
\end{aligned}$$

Then it follows from **(H1)** that

$$\begin{aligned}
&\mathbb{E} \left| \frac{1}{N-1} \sum_{j=1, j \neq i}^N \widehat{B}_t \bar{u}_{t-\theta}^j - m_0^\theta(t) \right|^2 \\
&\leq C_1 \left\{ \mathbb{E} \left| \frac{1}{N-1} \sum_{j=1, j \neq i}^N y_{t-\theta}^{j,1} - \mathbb{E} y_{t-\theta}^1 \right|^2 + \mathbb{E} \left| \frac{1}{N-1} \sum_{j=1, j \neq i}^N \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^j}}[y_t^{j,1}] - \mathbb{E} y_t^1 \right|^2 \right\},
\end{aligned}$$

where C_1 is a positive constant. Recall (9) that $y_t^{j,1} \in L^2_{\mathcal{F}_t^{W^j}}(-\theta, T + (\delta \vee \theta); \mathbb{R}^n)$. Thus $y_t^{j,1}$ is independent of $y_t^{k,1}$, for $j \neq k$, and we have

$$\mathbb{E}(y_{t-\theta}^{j,1} - \mathbb{E} y_{t-\theta}^1)(y_{t-\theta}^{k,1} - \mathbb{E} y_{t-\theta}^1) = 0$$

and

$$\mathbb{E}(\mathbb{E}^{\mathcal{F}_{t-\theta}^{W^j}}[y_t^{j,1}] - \mathbb{E} y_t^1)(\mathbb{E}^{\mathcal{F}_{t-\theta}^{W^k}}[y_t^{k,1}] - \mathbb{E} y_t^1) = 0.$$

Hence the result. \square

Lemma 4.2

$$\sup_{1 \leq i \leq N} \left[\sup_{0 \leq t \leq T} \mathbb{E} |\check{x}_t^i - \hat{x}_t^i|^2 \right] = O\left(\frac{1}{N}\right), \tag{19}$$

$$\sup_{1 \leq i \leq N} \left[\sup_{0 \leq t \leq T} \mathbb{E} |\check{x}_{t-\delta}^i - \hat{x}_{t-\delta}^i|^2 \right] = O\left(\frac{1}{N}\right). \tag{20}$$

Proof. For $\forall 1 \leq i \leq N$, by (1) and (3), we have

$$\begin{cases} d(\check{x}_t^i - \hat{x}_t^i) = \left[A_t(\check{x}_t^i - \hat{x}_t^i) + \tilde{A}_t(\check{x}_{t-\delta}^i - \hat{x}_{t-\delta}^i) + \frac{1}{N-1} \sum_{j=1, j \neq i}^N \hat{B}_t \bar{u}_{t-\theta}^j - m_0^\theta(t) \right] dt, & t \in [0, T], \\ \check{x}_0^i - \hat{x}_0^i = 0, & \check{x}_t^i - \hat{x}_t^i = 0, & t \in [-\delta, 0). \end{cases}$$

Taking integral from 0 to T , we get

$$\check{x}_T^i - \hat{x}_T^i = \int_0^T \left[A_s(\check{x}_s^i - \hat{x}_s^i) + \tilde{A}_s(\check{x}_{s-\delta}^i - \hat{x}_{s-\delta}^i) + \frac{1}{N-1} \sum_{j=1, j \neq i}^N \hat{B}_s \bar{u}_{s-\theta}^j - m_0^\theta(s) \right] ds.$$

Note that

$$\int_0^T \tilde{A}_s(\check{x}_{s-\delta}^i - \hat{x}_{s-\delta}^i) ds = \int_{-\delta}^{T-\delta} \tilde{A}_{s+\delta}(\check{x}_s^i - \hat{x}_s^i) ds = \int_0^{T-\delta} \tilde{A}_{s+\delta}(\check{x}_s^i - \hat{x}_s^i) ds.$$

By Lemma 4.1, **(H1)** and Gronwall's inequality, (19) is obtained.

In addition,

$$\sup_{0 \leq t \leq T} \mathbb{E} |\check{x}_{t-\delta}^i - \hat{x}_{t-\delta}^i|^2 = \sup_{0 \leq \tau \leq T-\delta} \mathbb{E} |\check{x}_\tau^i - \hat{x}_\tau^i|^2 \leq \sup_{0 \leq \tau \leq T} \mathbb{E} |\check{x}_\tau^i - \hat{x}_\tau^i|^2.$$

Then we get (20). \square

Lemma 4.3

$$\sup_{1 \leq i \leq N} \left[\sup_{0 \leq t \leq T} \mathbb{E} \left| |\check{x}_t^i|^2 - |\hat{x}_t^i|^2 \right| \right] = O\left(\frac{1}{\sqrt{N}}\right), \quad (21)$$

$$\sup_{1 \leq i \leq N} \left[\sup_{0 \leq t \leq T} \mathbb{E} \left| |\check{x}_{t-\delta}^i|^2 - |\hat{x}_{t-\delta}^i|^2 \right| \right] = O\left(\frac{1}{\sqrt{N}}\right), \quad (22)$$

$$\left| \mathcal{J}^i(\bar{u}_t^i, \bar{u}_t^{-i}) - J^i(\bar{u}_t^i) \right| = O\left(\frac{1}{\sqrt{N}}\right), \quad 1 \leq i \leq N. \quad (23)$$

Proof. For $\forall 1 \leq i \leq N$, it is easy to see $\sup_{0 \leq t \leq T} \mathbb{E} |\hat{x}_t^i|^2 < +\infty$, $\sup_{0 \leq t \leq T} \mathbb{E} |\hat{x}_{t-\delta}^i|^2 < +\infty$. Applying Cauchy-Schwarz inequality and (19), we have

$$\begin{aligned} & \sup_{0 \leq t \leq T} \mathbb{E} \left| |\check{x}_t^i|^2 - |\hat{x}_t^i|^2 \right| \\ & \leq \sup_{0 \leq t \leq T} \mathbb{E} |\check{x}_t^i - \hat{x}_t^i|^2 + 2 \left(\sup_{0 \leq t \leq T} \mathbb{E} |\hat{x}_t^i|^2 \right)^{\frac{1}{2}} \left(\sup_{0 \leq t \leq T} \mathbb{E} |\check{x}_t^i - \hat{x}_t^i|^2 \right)^{\frac{1}{2}} \\ & = O\left(\frac{1}{\sqrt{N}}\right). \end{aligned}$$

Similarly, (22) is obtained. Then noting **(H2)**, we have

$$\begin{aligned} & \left| \mathcal{J}^i(\bar{u}_t^i, \bar{u}_t^{-i}) - J^i(\bar{u}_t^i) \right| \\ & \leq C_2 \mathbb{E} \int_0^T (|\check{x}_t^i|^2 - |\hat{x}_t^i|^2| + |\check{x}_{t-\delta}^i|^2 - |\hat{x}_{t-\delta}^i|^2|) dt + C_2 \mathbb{E} |\check{x}_T^i|^2 - |\hat{x}_T^i|^2 \\ & = O\left(\frac{1}{\sqrt{N}}\right), \end{aligned}$$

which implies (23). Here, C_2 is a positive constant. \square

Until now, we have addressed some estimates of states and costs corresponding to control \bar{u}_t^i , $1 \leq i \leq N$. Next we will focus on the ϵ -Nash equilibrium for **(LD)**. For any fixed i , $1 \leq i \leq N$, consider a alternative control $u_t^i \in \mathcal{U}_i$ for \mathcal{A}_i and introduce the dynamics

$$\begin{cases} dl_t^i = \left[A_t l_t^i + \tilde{A}_t l_{t-\delta}^i + B_t u_t^i + \tilde{B}_t u_{t-\theta}^i + \frac{1}{N-1} \sum_{\kappa=1, \kappa \neq i}^N \hat{B}_t \bar{u}_{t-\theta}^\kappa \right] dt + \sigma_t dW_t^i + \sigma_t^0 dW_t^0, & t \in [0, T], \\ l_0^i = a, & l_t^i = \xi_t^i, \quad t \in [-\delta, 0), \quad u_t^i = \eta_t^i, \quad t \in [-\theta, 0), \end{cases} \quad (24)$$

whereas other players keep the control $\bar{u}_t^j, 1 \leq j \leq N, j \neq i$ i.e.,

$$\begin{cases} dl_t^j = \left[A_t l_t^j + \tilde{A}_t l_{t-\delta}^j + B_t \bar{u}_t^j + \tilde{B}_t \bar{u}_{t-\theta}^j + \frac{1}{N-1} \hat{B}_t \left(\sum_{\kappa=1, \kappa \neq i, j}^N \bar{u}_{t-\theta}^\kappa + u_{t-\theta}^i \right) \right] dt + \sigma_t dW_t^j + \sigma_t^0 dW_t^0, \\ l_0^j = a, & l_t^j = \xi_t^j, \quad t \in [-\delta, 0), \quad u_t^i = \eta_t^i, \quad t \in [-\theta, 0). \end{cases}$$

The dynamics of \mathcal{A}_i with respect to u_t^i for **(LDD)** is

$$\begin{cases} dp_t^i = \left[A_t p_t^i + \tilde{A}_t p_{t-\delta}^i + B_t u_t^i + \tilde{B}_t u_{t-\theta}^i + m_0^\theta(t) \right] dt + \sigma_t dW_t^i + \sigma_t^0 dW_t^0, & t \in [0, T], \\ p_0^i = a, & p_t^i = \xi_t^i, \quad t \in [-\delta, 0), \quad u_t^i = \eta_t^i, \quad t \in [-\theta, 0). \end{cases} \quad (25)$$

We have the following lemma.

Lemma 4.4

$$\sup_{1 \leq i \leq N} \left[\sup_{0 \leq t \leq T} \mathbb{E} |l_t^i - p_t^i|^2 \right] = O\left(\frac{1}{N}\right), \quad (26)$$

$$\sup_{1 \leq i \leq N} \left[\sup_{0 \leq t \leq T} \mathbb{E} |l_{t-\delta}^i - p_{t-\delta}^i|^2 \right] = O\left(\frac{1}{N}\right), \quad (27)$$

$$\sup_{1 \leq i \leq N} \left[\sup_{0 \leq t \leq T} \mathbb{E} \left| |l_t^i|^2 - |p_t^i|^2 \right| \right] = O\left(\frac{1}{\sqrt{N}}\right), \quad (28)$$

$$\sup_{1 \leq i \leq N} \left[\sup_{0 \leq t \leq T} \mathbb{E} \left| |l_{t-\delta}^i|^2 - |p_{t-\delta}^i|^2 \right| \right] = O\left(\frac{1}{\sqrt{N}}\right), \quad (29)$$

$$\left| \mathcal{J}^i(u_t^i, \bar{u}_t^{-i}) - J^i(u_t^i) \right| = O\left(\frac{1}{\sqrt{N}}\right), \quad 1 \leq i \leq N. \quad (30)$$

Proof. Using the same analysis to the proof of Lemma 4.2, by (24)-(25) and noting Lemma 4.1, we get (26) and (27). By virtue of (26) and (27), (28) and (29) follows by applying Cauchy-Schwarz inequality. Same to Lemma 4.3, (30) is obtained. \square

Proof of Theorem 4.1: Now, we consider the ϵ -Nash equilibrium for $\mathcal{A}_i, 1 \leq i \leq N$. It follows from (23) and (30) that

$$\begin{aligned}\mathcal{J}^i(\bar{u}_t^i, \bar{u}_t^{-i}) &= J^i(\bar{u}_t^i) + O\left(\frac{1}{\sqrt{N}}\right) \\ &\leq J^i(u_t^i) + O\left(\frac{1}{\sqrt{N}}\right) \\ &= \mathcal{J}^i(u_t^i, \bar{u}_t^{-i}) + O\left(\frac{1}{\sqrt{N}}\right).\end{aligned}$$

Thus, Theorem 4.1 follows by taking $\epsilon = O\left(\frac{1}{\sqrt{N}}\right)$.

5 Special cases

In this section, we will study some special cases to show the essence of MFG problem with delay.

Case I: In this case, we will give the “closed-loop” form of the ϵ -Nash equilibrium. For simplicity, let $\tilde{A}_t = \tilde{B}_t = 0$ in system (1), then we study the following system,

$$\begin{cases} dx_t^i = \left[A_t x_t^i + B_t u_t^i + \frac{1}{N-1} \sum_{j=1, j \neq i}^N \hat{B}_t u_{t-\theta}^j \right] dt + \sigma_t dW_t^i + \sigma_t^0 dW_t^0, \quad t \in [0, T], \\ x_0 = a, \end{cases} \quad (31)$$

and the cost functional is still (2).

Now, we consider the following FBSDE

$$\begin{cases} d\bar{x}_t^i = \left[A_t \bar{x}_t^i - B_t(N_t + \tilde{N}_{t+\theta})^{-1} B_t^\top \bar{y}_t^i + m_0^\theta(t) \right] dt + \sigma_t dW_t^i + \sigma_t^0 dW_t^0, \\ d\bar{y}_t^i = - \left[A_t^\top \bar{y}_t^i + (R_t + \tilde{R}_{t+\delta}) \bar{x}_t^i \right] dt + \bar{z}_t^i dW_t^i + \bar{z}_t^0 dW_t^0, \quad t \in [0, T], \\ \bar{x}_0^i = a, \\ \bar{y}_T^i = M \bar{x}_T^i, \end{cases} \quad (32)$$

In system (32), we could deduce the $m_0^\theta(t)$ as follows,

$$\begin{aligned}m_0^\theta(t) &= \lim_{N \rightarrow \infty} \hat{B}_t \frac{1}{N-1} \sum_{j=1, j \neq i}^N \bar{u}_{t-\theta}^j \\ &= -\hat{B}_t(N_{t-\theta} + \tilde{N}_t)^{-1} B_{t-\theta}^\top \lim_{N \rightarrow \infty} \frac{1}{N-1} \sum_{j=1, j \neq i}^N \bar{y}_{t-\theta}^j \\ &= -\hat{B}_t(N_{t-\theta} + \tilde{N}_t)^{-1} B_{t-\theta}^\top \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^0}}[\bar{y}_{t-\theta}]\end{aligned}$$

where \tilde{y}_t satisfies the following FBSDDE,

$$\begin{cases} d\tilde{x}_t = \left[A_t\tilde{x}_t - B_t(N_t + \tilde{N}_{t+\theta})^{-1}B_t^\top \tilde{y}_t - \hat{B}_t(N_{t-\theta} + \tilde{N}_t)^{-1}B_{t-\theta}^\top \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^0}}[\tilde{y}_{t-\theta}] \right] dt + \sigma_t dW_t \\ \quad + \sigma_t^0 dW_t^0, \\ d\tilde{y}_t = -\left[A_t^\top \tilde{y}_t + (R_t + \tilde{R}_{t+\delta})\tilde{x}_t \right] dt + \tilde{z}_t dW_t + \tilde{z}_t^0 dW_t^0, \quad t \in [0, T], \\ \tilde{x}_0 = a, \\ \tilde{y}_T = M\tilde{x}_T, \end{cases} \quad (33)$$

where W_t and W_t^i are independent and identically distributed. According to the Appendix in [9], (33) has a unique solution $(\tilde{x}_t, \tilde{y}_t, \tilde{z}_t, \tilde{z}_t^0)$.

Then, FBSDDE (32) could be rewritten as follows

$$\begin{cases} d\bar{x}_t^i = \left[A_t\bar{x}_t^i - B_t(N_t + \tilde{N}_{t+\theta})^{-1}B_t^\top \bar{y}_t^i - \hat{B}_t(N_{t-\theta} + \tilde{N}_t)^{-1}B_{t-\theta}^\top \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^0}}[\bar{y}_{t-\theta}] \right] dt \\ \quad + \sigma_t dW_t^i + \sigma_t^0 dW_t^0, \\ d\bar{y}_t^i = -\left[A_t^\top \bar{y}_t^i + (R_t + \tilde{R}_{t+\delta})\bar{x}_t^i \right] dt + \bar{z}_t^i dW_t^i + \bar{z}_t^0 dW_t^0, \quad t \in [0, T], \\ \bar{x}_0^i = a, \\ \bar{y}_T^i = M\bar{x}_T^i. \end{cases} \quad (34)$$

FBSDE (34) could be decoupled by the following Riccati equation and ordinary differential equation

$$\begin{cases} \dot{P}_t + P_t A_t + A_t^\top P_t + R_t + \tilde{R}_{t+\delta} - P_t B_t (N_t + \tilde{N}_{t+\theta})^{-1} B_t^\top P_t = 0, \\ P_T = M, \end{cases}$$

and

$$\begin{cases} \dot{\phi}_t + [A_t - B_t(N_t + \tilde{N}_{t+\theta})^{-1}B_t^\top P_t]\phi_t - P_t \hat{B}_t(N_{t-\theta} + \tilde{N}_t)^{-1}B_{t-\theta}^\top \mathbb{E}^{\mathcal{F}_{t-\theta}^{W^0}}[\tilde{y}_{t-\theta}] = 0, \\ \phi_T = 0. \end{cases}$$

We obtain the optimal feedback is

$$\bar{u}_t^i = -(N_t + \tilde{N}_{t+\theta})^{-1}B_t^\top (P_t \bar{x}_t^i + \phi_t).$$

From Theorem 4.1, we claim that $(\bar{u}_t^1, \bar{u}_t^2, \dots, \bar{u}_t^N)$ is the ϵ -Nash equilibrium of the Problem (31), (2).

Case II: Now, we consider another special case. Let $A_t = B_t = 0$ in system (1), moreover, we assume $\delta = \theta$, then we study the following system

$$\begin{cases} dx_t^i = \left[\tilde{A}_t x_{t-\delta}^i + \tilde{B}_t u_{t-\delta}^i + \frac{1}{N-1} \sum_{j=1, j \neq i}^N \hat{B}_t u_{t-\delta}^j \right] dt + \sigma_t dW_t^i + \sigma_t^0 dW_t^0, \quad t \in [0, T], \\ x_0^i = a, \quad x_t^i = \xi_t^i, \quad t \in [-\delta, 0), \quad u_t^i = \eta_t^i, \quad t \in [-\theta, 0), \end{cases} \quad (35)$$

and the cost functional is

$$\mathcal{J}^i(u_t^i, u_t^{-i}) = \frac{1}{2} \mathbb{E} \int_0^T [N_t (u_t^i)^2 + \tilde{N}_t (u_{t-\theta}^i)^2] dt + M x_T^i.$$

We will consider the following system instead of (35)

$$\begin{cases} dx_t^i = [\tilde{A}_t x_{t-\delta}^i + \tilde{B}_t u_{t-\delta}^i + m_0^\delta(t)] dt + \sigma_t dW_t^i + \sigma_t^0 dW_t^0, & t \in [0, T], \\ x_0^i = a, & x_t^i = \xi_t^i, & t \in [-\delta, 0), & u_t^i = \eta_t^i, & t \in [-\theta, 0). \end{cases} \quad (36)$$

and the adjoint equation is

$$\begin{cases} dy_t^i = -\tilde{A}_{t+\delta} \mathbb{E}^{\mathcal{F}_t^i} [y_{t+\delta}^i] dt + z_t^i dW_t^i + z_t^0 dW_t^0, & t \in [0, T], \\ y_T^i = -M, \\ y_t^i = 0, & t \in (T, T + (\delta \vee \theta)]. \end{cases} \quad (37)$$

We could solve (37) explicitly by applying the method in [24], which can also be found in [19].

(i) When $t \in [T - \delta, T]$, the ABSDE (37) becomes

$$\bar{y}_t^i = -M - \int_t^T \bar{z}_s^i dW_s^i - \int_t^T \bar{z}_s^0 dW_s^0, \quad t \in [T - \delta, T]$$

We could solve

$$\bar{y}_t = -M, \quad \bar{z}_t^i = 0, \quad \bar{z}_t^0 = 0, \quad t \in [T - \delta, T].$$

(ii) If we solve (37) on the interval $[T - k\delta, T - (k - 1)\delta]$ ($k = 1, 2, 3 \dots$), and the solution $\{(\bar{y}_t^i, \bar{z}_t^i, \bar{z}_t^0); T - k\delta \leq t \leq T - (k - 1)\delta\}$ is Malliavin differentiable, then we could solve the (37) on the next interval $[T - (k + 1)\delta, T - k\delta]$,

$$\bar{y}_t^i = \mathbb{E}[\bar{y}_{T-k\delta}] + \int_t^{T-k\delta} \tilde{A}_s \mathbb{E}^{\mathcal{F}_s^i} [\bar{y}_{s+\delta}] ds,$$

and

$$\bar{z}_t^i = 0, \quad \bar{z}_t^0 = 0, \quad t \in [T - (k + 1)\delta, T - k\delta].$$

The optimal control is

$$\bar{u}_t^i = -(N_t + \tilde{N}_{t+\delta})^{-1} \tilde{B}_{t+\delta} \mathbb{E}^{\mathcal{F}_t^i} [\bar{y}_{t+\delta}].$$

So the ϵ -Nash equilibrium is $(\bar{u}_t^1, \bar{u}_t^2, \dots, \bar{u}_t^N)$.

Next, we consider a special case that the coefficients are all constants: $\tilde{A}_t = \tilde{A}$, $\tilde{B}_t = \tilde{B}$, $M = 1$, $N_t = N$, $\tilde{N}_t = \tilde{N}$, then the solution of (37) as follows,

$$\begin{aligned} \bar{y}_{t+\delta}^i &= 0, \quad \bar{z}_t^i = 0, \quad \bar{z}_t^0 = 0, & t \in [T - \delta, T]; \\ \bar{y}_{t+\delta}^i &= -1, \quad \bar{z}_t^i = 0, \quad \bar{z}_t^0 = 0, & t \in [T - 2\delta, T - \delta]; \\ \bar{y}_{t+\delta}^i &= -1 - \tilde{A}(T - \delta - t), \quad \bar{z}_t^i = 0, \quad \bar{z}_t^0 = 0, & t \in [T - 3\delta, T - 2\delta]; \\ \bar{y}_{t+\delta}^i &= -1 - \tilde{A}\delta - \tilde{A}(T - 2\delta - t)[1 + \frac{1}{2}\tilde{A}(T - 2\delta - t)], \quad \bar{z}_t^i = 0, \quad \bar{z}_t^0 = 0, & t \in [T - 4\delta, T - 3\delta]; \\ &\dots \end{aligned}$$

Then, ϵ -Nash equilibrium is $(\bar{u}_t^1, \bar{u}_t^2, \dots, \bar{u}_t^N)$, where

$$\bar{u}_t^i = -\frac{\tilde{B}}{N + \tilde{N}} \bar{y}_{t+\delta}^i.$$

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